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SPHERICAL GEOMETRY.

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LECTURE IV. THE PARTITION OF THE SURFACE.

The theorems which have been derived in the foregoing lectures have been concerned with that which obtains at points of the surface or along lines on the surface. Little has been said of the characteristics of the surface taken as a whole. It will be remembered that in the definition of a triangle a caution not to regard the triangle as a portion of the surface was especially mentioned. It would be equally improper to speak of the interior or exterior of the triangle: for no demonstration or axiom has yet been given to show that the triangle has properties such as are implied by the use of these words. With this deficiency is closely connected (though it might not appear so at first) the rudimentary state of our knowledge of angles, due as has been seen to the fact that we are not assured of any natural arrangement among the directions issuing from a point. This renders it difficult if not impossible to proceed to right angles and the relations of symmetry. To set aside both these limitations we shall now investigate the statement: A line divides the surface of the sphere into two separate parts.

Describe a line l on the surface. Relative to this line two points A and B may have one of two positions. If the line joining A and B cuts the line l in L and M both of which lie within or without the segment AB the two points A and B may be said to lie on the same side of the line l: but if the points L and M are so situated that one lies within the segment AB and the other without that segment, then it would be natural to say that A and B lie on opposite sides of the line l. Thus the points of the surface may be assorted into three classes of which the first contains all the points on the line l, the second all the points

which lie on the same side of l as A, and the third all points which lie on the opposite side of l from A. Choose another point, say B, and again divide the points of the surface into three similar classes, points on l, points on the same side of l as B, and points on the opposite side of l from B. Now it is a fundamental property of our conception of divisibility of a surface into parts that if A and B are two points of the same part then any point which is in the same part with A is in the same part with B and vice versa. If there were laid down a definition by virtue of which the surface could be said to be divided into parts and if later it became apparent that in accordance with the definition a point C might be in the same part with A and B without necessitating that A and B were in the same part, we should be forced to conclude either that our definition was useless or that the phrase "divided into parts" had been applied in a sense quite different from the usual. With these preliminary remarks on the meaning of division into parts and its significance we may lay down

Axiom VI. *A line separates the surface of a sphere into two parts: these parts being such that if two points B and C are in the same part with A, then they are in the same part with each other.*

Theorem 14. *Any two lines intersect in two points which are situated on opposite sides of any arbitrary line not passing through their points of intersection.*

Theorem 15. *If the surface is divided into two parts by the line l, then the segment of a line which connects any point of one of those parts to any point of the line l lies wholly in that part.*

Theorem 16. *If a line cuts one side of a proper triangle it must also cut one of the other sides or pass through their intersection.*

The proof of the first two of these theorems is left to the reader, that of the last is as follows. Let the line l cut the side AB of the triangle ABC. As the triangle is a proper triangle the side AB is not so great as a semi-line and the line l cannot cut this side in a second point. Hence the points A and B are situated on opposite sides of the line l. If l cuts neither AC nor BC nor passes through the point C, and points A and C lie upon the same side of l, and likewise the points B and C. This is contrary to the fact that A and B are on opposite sides of the line and hence the theorem is proved.

As an immediate corollary it follows that a line cannot cut all three sides of a triangle.

Theorem 17. *Two semi-lines OAO' and OBO' which connect a pair of antipodal points divide the surface into two parts.*

From Theorem 14 it follows that any line drawn on the surface cuts the two semi-lines in question or passes through their points of intersection O and O'. Draw on the surface any line l which does not pass through the points of intersection of the semi-lines and let it cut them in the points A and B. By the points A and B the line l is divided into two segments of which one is less than a semi-line and the other greater, except in the special case when the two semi-lines happen to form one complete line. Let C be any point of the proper segment AB. From Theorem 15 it follows that the segment joining C to any point

of OAO', or OAO' produced so as to form a complete line, is less than a semi-line. By so much the more is the segment which joins C to any point of OBO' or OAO' less than a semi-line. Consider the class of points D such that the proper segment CD does not cut either OAO' or OBO'. Consider also the class of points E such that the segment CE cuts one of the semi-lines or passes through the vertices O and O'. Every point of the surface, except such as are situated on the given semi-lines, occurs once and but once in one of these sets. Moreover from Theorem 16 it is evident that any two of the points D may be connected by a segment which does not cut either of the semi-lines and that any two of the points E may also be so connected (although in this case the segment cannot always be taken less than a semi-line) and that no point D can be connected with a point E by a segment which does not cut one of these given semi-lines. Hence the theorem is proved.

It has been seen that the points in one of the portions of the surface have the property that any segment drawn through them and intercepted by the given semi-lines are all less than a semi-line. This portion of the surface may naturally be said to be the interior of the figure and may be said to be enclosed by the broken line $OAO'BO$. We should note at this point that we have actually defined the words "interior" and "enclosed by." Our definitions are based upon properties of our figures proved from the axioms laid down. This is not usually done. It is customary to assume all such properties. We may proceed along the same lines to the following theorems.

Theorem 18. *A proper triangle divides the surface of the sphere into two parts, an interior and an exterior, such that a segment drawn through any point of the interior and intercepted by the side of the triangle is less than a semi-line, and any two points on the exterior may be connected by a segment which does not cut any side of the triangle.*

Theorem 19. *If segments are drawn to join two vertices of a triangle to points of the opposite sides they meet within the triangle.*

The proof of the former of these theorems is left to the reader, that of the latter depends on Theorem 16. Let ABC be the triangle and A', B' two points of the sides opposite A, B, respectively. Consider the triangle ACA'. The line BB' cuts the side AC by hypothesis and cannot cut the side CA' because BC is less than a semi-line. Hence it cuts AA' and the theorem is proved. There is considerable interest attaching to this theorem owing to the fact that it is often taken to be the definition of a triangle.*

A set of segments AB, BC, CD, DE, ..., is said to form a broken line. The broken line is *simple* when it does not pass twice through the same point. It is *closed* when the last point of the last segment coincides with the first point of the first segment. It if *convex* when no line on the surface cuts it in more than two points; otherwise it is *re-entrant*. A closed simple broken line may be called a *polygon*.

*See Schur, Ueber die Grundlagen der Geometrie, *Mathematische Annalen*, Vol. 55, p. 268, postulate 7; and Peano, Sui fondamenti della Geometria, *Rivista di Matematica*, Vol. 4, p. 55, et seq.

Theorem 20. *It is impossible to draw a triangle two of whose sides are semi-lines and if two of the sides are greater than semi-lines the triangle is not simple.*

Theorem 21. *If one side of a simple triangle is greater than a semi-line the triangle is improper and re-entrant.*

In the latter theorem let ABC be the triangle and AC the side which is not proper. Form the triangle which has for sides AB and BC and the proper segment drawn between A and C. This is a proper triangle and by Theorem 16 any line which cuts the sides AB and BC cannot cut the third side, the proper segment AC. Hence the line must cut the improper segment AC in two points and the given triangle is reëntrant. This is the reason that in working with triangles on the sphere the triangle is understood to be proper unless the contrary is specifically stated.

Theorem 22. *If two points of a broken line are on opposite sides of a line l, the broken line cuts the line l in at least one point and in at least two points if the broken line is closed.*

Let $ABC.....EFG$ be the broken line and P, Q two of its points which are situated on opposite sides of the line l. These points lie on certain of the sides of the broken line, say BC and EF. Consider the points $P, C, D, \ldots, E, Q$. Separate these points into three classes such that those in the first lie on the same side as P, those in the second on the same side as Q, and those in the third on the line itself. In case there is a point belonging to the third class the theorem is proved for the broken line cuts the line l at this point. In case there is no such point examine the sequence of points $P, C, D, \ldots, E, Q$. Since the first and last of this sequence are situated on opposite sides of the line l there must be two successive points of the sequence which lie on opposite sides of l, and the segment which joins these points must cut l. Hence the broken line cuts l. If the broken line be closed, as $ABC.....FGA$, it may be divided into two broken lines $PCD.....EQ$ and $QFGABP$ to each of which the foregoing reasoning applies. Hence the broken line cuts the line l in at least two points.

Theorem 23. *A convex polygon lies entirely on one side of or along the line formed by producing a side of the polygon so as to form a complete line; entirely within or along the contour of the figure formed by producing two of the sides until they intersect; and entirely within or along the contour of the triangle formed by producing three of the sides until they intersect.*

Theorem 24. *A convex polygon divides the surface into two portions, an interior and an exterior, such that any two points of the interior may be joined by a proper segment which does not cut the polygon, any two points of the exterior may be joined by a segment not necessarily proper which does not cut the polygon, and no point of the interior can be joined to any point of the exterior by a segment which does not cut the polygon.*

These theorems are demonstrated by methods so like those already given that the details may be omitted. In case of reëntrant polygons the theorems become somewhat more complicated and are of decidedly less value in elementary work. The following theorems may be stated.

Theorem 25. *Any polygon of a finite number of sides divides the surface into portions, an interior and an exterior, such that any two points of the interior or of the exterior may be joined by a broken line which does not cut the polygon, and no point of the interior can be joined to any point of the exterior by a broken line which does not cut the polygon.*

It was stated at the outset of this lecture that intimately associated with the division of the surface was the troublesome question of the arrangement of the directions issuing from a point. The solution of the difficulty depends on Theorem 17 by means of which it will be possible to prove

Theorem 26. *The directions issuing from a point may be arranged in a natural order, with respect to all lines which do not pass through the point, by associating each direction with the point in which it cuts a given line not passing through the point and by assigning to the directions the order which the points associated with them have.**

Let O be the point from which the directions issue and O' the antipodal point. The directions from O and O' each cut in one and only one point any arbitrary line l not passing through O and conversely to each point of the line corresponds one and only one direction obtainable by joining the point to O. Set up a similar correspondence between the directions issuing from O and the points of a second line l'. As the points of the lines l and l' are associated in a one to one manner with the directions issuing from the point O, they must be associated in a one to one manner with each other. Therefore, if a point describes one line and passes once and only once over each point of the line, then the corresponding point of the other line will describe the line in such a manner as to pass once and only once over each point. It remains to show that the order of description is the same. We shall show that the words "lie between" have the same significance for both lines. Let $OAA'O'$ be a direction which cuts l in A and l' in A'. Let B, C, D be three points of l and B', C', D' the three corresponding points of l'. If C lies in that segment BD which does not contain A, it will be said to be between B and D; and similarly for the corresponding points. The semi-lines $OBB'O'$ and $ODD'O'$ divide the surface into two parts in one of which A lies and in the other of which C is found if it be between B and D. The semi-line $OCC'O'$ lies in the same portion of the surface as C does. Hence C and C' lie in the same part with each other. But A and C are in different parts and hence A' and C' are in different parts. Therefore C' lies between B' and D'. Hence l and l' are described in the same or in the opposite order. As the order on the two lines is independent we may say that the order ACD on one is the same as the order $A'C'D'$ on the other, and the theorem is proved.

Theorem 27. *In case a convex polygon (including the proper triangle and the figure formed by two semi-lines which connect a pair of antipodal points) is subjected to a motion, the resulting figure is a convex polygon of which the interior corresponds to the interior of the original polygon.*

*The directions being related in a one to one manner to the points on a line and possessing the same order may be said to form a continuum just as the points on a line do.

Theorem 28. *If the point O remains fixed during a motion the directions issuing from O change their positions without suffering a change in order.*

The former theorem is left to the reader; the latter may be proved as follows. Let OAO' and OBO' be two directions between which the direction OCO' lies. Let the three directions be moved into $OA'O'$, $OB'O$, and $OC'O'$, respectively. The region enclosed by OAO' and OBO' is, by the foregoing theorem, moved into the region enclosed by $OA'O'$ and $OB'O'$. Hence if OC lies in the original region, OC' lies in the transformed region. The words "lie between" have, therefore, the same significance before and after the motion. Hence the order is not changed or it is reversed.

To rule out the second supposition consider a line l on the surface and let the directions issuing from O be associated with the points on l as described under Theorem 26. Let OA be moved into OA' and suppose that the intersections of these directions with the line l be $\bar{A}$ and $\bar{A}'$. Let OX be a variable direction issuing from O and cutting l in $\bar{X}$. Now cause $\bar{X}$ to describe continuously the line l from $\bar{A}$ to $\bar{A}'$. If OX' be the direction which after the motion corresponds to OX the point $\bar{X}'$ will describe the line l from $\bar{A}'$ towards $\bar{A}$. Hence at some position $\bar{X}$ and $\bar{X}'$ will coincide and to this point will correspond a fixed direction issuing from O.* This is impossible for then there would be no motion (Theorem 1). Hence the order of the directions issuing from a point cannot be changed by a motion. The theorem is proved. And from this point we can proceed in the next lecture to develop further the theory of angles and in particular to the establishment of the existence and properties of right angles.

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*This conclusion is based on two steps which we shall not prove here but which ought to be mentioned and may be left to the reader to demonstrate. First, if one of two rigidly connected directions OA and OB is moved continuously about the point O, the other moves continuously about that point. Second, if two points X and Y, starting from two points A and B, move continuously in opposite directions until X falls on B, then they must have coincided at some point.